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Plastic Buckling of Cylindrical Shells for Cases of  
Non-homogeneous Stress Distribution

Contract No. DA-30-115-ORD-718 of Office of Ordnance Research,  
U. S. Army, Final Report

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Historical Synopsis. After the writer developed his theory on the plastic buckling of plates and shells [1]<sup>1</sup>, he applied it to cases of homogeneous stress distribution for the most important cases of loading and boundary conditions and showed its excellent agreement with test results [2], [3], [4]. He then obtained a contract from the Office of Ordnance Research, U. S. Army, for applying it to several cases of non-homogeneous stress distribution in plates. The results of this work was published in three papers [5], [6], [7].

In an earlier paper [3], he had applied his theory to the plastic buckling of a cylindrical shell under axial compressive load. As also explained in Reference 4 such a cylindrical shell, if the tangent modulus  $E_t$  is zero, has the same plastic buckling stress as a wide column with the same half wave length of buckles. In the case of bending of such a shell, the plastic fibers are supported laterally by the elastic ones, so that for low values of  $E_t$  the edge buckling stress can be expected to be much higher than for pure compression with the same tangent modulus  $E_t$  and secant modulus  $E_s$ . Therefore, the present contract was applied for, in order to compute plastic buckling stresses for several cases of plastic buckling of cylindrical shells subjected to bending and subsequently derive from the results of a simple design formula for comput-

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<sup>1</sup> Numbers in brackets designate References at end of report.

ing the plastic reduction coefficient to be applied to the elastic buckling stress.

Method Employed. From Reference 3 the differential equations for the

plastic buckling of a cylindrical shell under axial compressive stresses  $\sigma_x$  are

$$A \frac{\partial^2 u}{\partial x^2} + \frac{B+2F}{a} \frac{\partial^2 v}{\partial x \partial \theta} - \frac{B}{a} \frac{\partial w}{\partial x} + \frac{F}{a^2} \frac{\partial^2 u}{\partial \theta^2} = 0 \quad (1)$$

$$\begin{aligned} & (B+2F) \frac{\partial^2 u}{\partial x \partial \theta} + a F \frac{\partial^2 v}{\partial x^2} + \frac{D}{a} \left( \frac{\partial^2 v}{\partial \theta^2} - \frac{\partial w}{\partial \theta} \right) + \\ & \alpha \left[ \frac{D}{a} \left( \frac{\partial^2 v}{\partial \theta^2} + \frac{\partial^3 w}{\partial \theta^3} \right) + a (B+2F) \frac{\partial^3 w}{\partial x^2 \partial \theta} + 2 a F \frac{\partial^2 v}{\partial x^2} \right] \\ & - \frac{\alpha t \sigma_x}{E t} \frac{\partial^2 v}{\partial x^2} = 0 \end{aligned} \quad (2)$$

$$\begin{aligned} & - \frac{\alpha t \sigma_x}{E t} \frac{\partial^2 w}{\partial x^2} + B \frac{\partial u}{\partial x} + \frac{D}{a} \frac{\partial v}{\partial \theta} - \frac{D}{a} w - \alpha \left[ a (B+4F) \frac{\partial^3 v}{\partial x^2 \partial \theta} \right. \\ & \left. + \frac{D}{a} \frac{\partial^3 v}{\partial \theta^3} + a^3 A \frac{\partial^4 w}{\partial x^4} + 2 a (B+2F) \frac{\partial^4 w}{\partial x^2 \partial \theta^2} + \frac{D}{a} \frac{\partial^4 w}{\partial \theta^4} \right] = 0 \end{aligned} \quad (3)$$

$$\alpha = t^2 / (12 a^2)$$

where  $u$ ,  $v$ , and  $w$  are the displacements in the axial ( $x$ -), tangential ( $a\theta$ -), and radial ( $z$ -) directions, respectively,  $a$  and  $t$  are the shell radius and thickness, and  $A$ ,  $B$ ,  $D$ , and  $F$  are plastic reduction coefficients, which are, for example, given in reference 4 as functions of the tangent and secant moduli and of Poisson's ratio in the elastic range. These equations may be reduced to a partial differential equation in  $w$  only, namely

$$L_1^4 L_2^4 w + \frac{12}{a^2 t^2} (AD - B^2) F \frac{\partial^4 w}{\partial x^4} + \frac{12 \sigma_x}{t^2 E} L_1^4 \frac{\partial^2 w}{\partial x^2} = 0 \quad (4)$$

where

$$L_1^4 = AF \frac{\partial^4}{\partial x^4} + [AD - B^2 - 2BF] \frac{\partial^4}{a^2 \partial x^2 \partial \theta^2} + DF \frac{\partial^4}{a^4 \partial \theta^4}$$

$$L_2^4 = A \frac{\partial^4}{\partial x^4} + 2(B + 2F) \frac{\partial^4}{a^2 \partial x^2 \partial \theta^2} + D \frac{\partial^4}{a^4 \partial \theta^4}$$

Eq. (4) was reduced to an ordinary differential equation by the substitution

$$w = Y \sin \frac{\pi}{l} x$$

where Y is a function of  $y = a\theta$  only and  $l$  is the half wave length of buckles in the axial direction. Dividing the circumference of the shell into 32 spacings this ordinary differential equation was written as a set of 17 (due to symmetry) finite difference equations, using second order finite differences, as were also used for the case of buckling of a plate under plastic bending in its plane (reference 5). This results in the following difference equation for an arbitrary point k:

$$\begin{aligned} & -C_1(Y_{k-5} + Y_{k+5}) + C_2(Y_{k-4} + Y_{k+4}) - C_3(Y_{k-3} + Y_{k+3}) \\ & + C_4(Y_{k-2} + Y_{k+2}) - C_5(Y_{k-1} + Y_{k+1}) + C_6 Y_k = 0 \end{aligned} \quad (5)$$

where

$$C_1 = 4 F_1$$

$$C_2 = 52 F_1 + 3 h^2 F_2$$

$$C_3 = 276 F_1 + 36 h^2 F_2 + 2 h^4 F_3$$

$$C_4 = 816 F_1 + 156 h^2 F_2 + 24 h^4 F_3 + h^6 F_4$$

$$C_5 = 1512 F_1 + 348 h^2 F_2 + 78 h^4 F_3 + 16 h^4 F_4$$

$$C_6 = 1848 F_1 + 450 h^2 F_2 + 112 h^4 F_3 + 30 h^6 F_4 + 12 h^8 F_5$$

and

$$F_1 = D^2 F$$

$$F_2 = (AD^2 - B^2 D + 4 DF^2) \lambda^2$$

$$F_3 = 2(3ADF + ABD - 4B^2 F - 4BF^2 - B^3) \lambda^4 - \rho \varphi^2 DF \lambda^2$$

$$F_4 = A(4F^2 - B^2 + AD) \lambda^6 - \rho \varphi^2 (AD - B^2 - 2BF) \lambda^4$$

$$F_5 = A^2 F \lambda^8 + \frac{12}{a^2 t^2} (AD - B^2) F \lambda^4 - \rho \varphi^2 A F \lambda^6$$

In here  $h$  is the spacing  $\pi a/16$  of the pivotal points on the circumference,

$\lambda = \pi/\ell$ , where  $\ell$  is the half wave length of buckles in the axial (x-)

direction. Further  $\varphi^2 = 12 \sigma_0 / Et^2$  where  $\sigma_0$  is a reference stress, and

$\rho = \sigma_x / \sigma_0$ . The 17 equations (5) lead to a 17th order determinant which

has to be equal to zero from which the edge buckling stress can be calculated.

Cases 1 and 2 were calculated, where the plastic range extends to

$(5/16)\pi$  and  $(3/16)\pi$  from top and bottom, respectively. For the plastic range

several constant values of the tangent modulus were assumed and also, to establish the optimum half wave length  $l$ , several ratios  $l/a$ . The difficulties with this project were (1) to find graduate students that were interested in computing the 17th order determinants that, at the same time, worked sufficiently accurate, (2) the fact that still inaccuracies occurred, which, in most cases, leads to non-converging solutions of the determinant, (3) the students cannot spend very much time on this work, so that it progresses slowly and, after the student graduates, a new one has to be found, (4) The difficulties of finding the numerical errors, and (5) the evaluation of the determinants for finding the lowest eigenvalue.

Only the last difficulty was finally solved by having the determinants evaluated at Bell Aircraft Company in Buffalo, New York, on their 704 IBM machine. The result was that only the following cases could be evaluated, where the last column gives the edge stresses  $\sigma_e$  at incipient buckling. It was assumed that  $a/t = 1000/\sqrt{12}$ .

Table I

$E_t/E$	Case	$l/a$	$\sigma_e/E$
0	1	.5	.00000978
0	1	.7	.0000505
0	2	.5	.0000373
0	2	.7	.0001137
.9	1	.5	.00206
.9	1	.7	.001905
.9	1	.9	.001984
.9	2	.5	.002093
.9	2	.7	.001944
.9	2	.9	.002061
1		.7	.002032

In order to establish a general design formula also values for  $E_t/E=0$ ,  $l/a = .3$  and several cases for  $E_t/E$  of .1 and .5 have to be evaluated. Determinants for most of these cases were computed (some after the contract was finished at my own expense) but they will still have to be checked for accuracy. This will be done if time for it is available.

The numerical results obtained, as given in Table I, are not sufficient for deriving conclusions for other cases.



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